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THE CORRESPONDENCE BETWEEN CAVITIES AND RIGID INCLUSIONS IN THREE-DIMENSIONAL ELASTICITY AND THE COSSERAT SPECTRUM

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Abstract—The Dundurs' correspondence in plane elasticity between the stress fields of cavities and rigid inclusions in the limit as $\lambda+2\mu \to 0$ (or $k \to -1$) holds true also in three-dimensional elasticity, but only for dilatation constant or linear function of position. In general, in the limit $\lambda+2\mu \to 0$, the normal traction between rigid inclusion and matrix vanishes, and the shear traction is $\mathbf{T}^n = 2\mu(\mathbf{\Omega}-\boldsymbol{\omega})|_{k=-1} \times \mathbf{n}$. An example of a rigid spherical inclusion in a sphere under pressure is presented. A proof for the existence of the limit as $\lambda+2\mu \to 0$ (when ellipticity fails) is also presented based on the properties of the Cossérat spectrum.

INTRODUCTION

The correspondence in two-dimensional elasticity between the stress fields of cavities and rigid inclusions as the limit of the Kolosov constant $k \to -1$ (or for plane strain the Poisson's ratio $\nu \to 1$), has been discovered by Dundurs (1989) with the value of the traction on the cavities determined by Markenscoff (1993).

In this paper, the proof of Markenscoff (1993) is extended to three-dimensional elasticity. It is shown that equivalence holds only when the dilational field of the inclusion is a constant or linear function of position. However, it is shown that in three, like in two dimensions, the normal component of the traction at the interface between the rigid inclusion and the matrix vanishes as $k \to -1$. The dependence of the limit $k \to -1$ of the shear traction on the cavity on the relative rotation between the inclusion and the matrix is, in three dimensions, the exact generalization of the two-dimensional one: $\mathbf{T} = 2\mu(\mathbf{\Omega}-\boldsymbol{\omega})$ (Markenscoff, 1993). The existence of the limit $k \to -1$ is proved on the basis of the proof that the eigenvalue of the Cossérat spectrum as $k \to -1$ is isolated.

CORRESPONDENCE BETWEEN CAVITIES AND RIGID INCLUSIONS IN THREE DIMENSIONS

In two dimensions, the traction boundary-value problem of elasticity does not depend on the elastic constants of the material if the Michell conditions are satisfied. It was shown (Markenscoff, 1993) that if the limit of the stress of the inclusion boundary-value problem, which depends on the elastic constants, exists as $k \to -1$, then it constitutes the solution of the traction boundary-value problem for the cavity problem, since it satisfies equilibrium and the boundary conditions and, by the uniqueness theorem for the traction boundary-value problem, exists and is unique.

In three dimensions, the solution of the traction boundary-value-problem of elasticity does depend on Poisson's ratio since the Beltrami–Michell compatibility conditions for the stress depend on Poisson's ratio, so that a similar argument for correspondence between cavities and rigid inclusions cannot in principle be made. However, the Beltrami–Michell compatibility conditions are independent of v if the dilation is a constant or a linear function of the position. In this case we can show that the same line of proof goes through for the correspondence.

The stress–strain relations

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{1}$$

yield for $\lambda + 2\mu = 0$

$$\begin{aligned} \tau_{11} &= -\mu(u_{2,2}+u_{3,3}) \quad & \tau_{12} &= \mu(u_{1,2}+u_{2,1}) \\ \tau_{22} &= -\mu(u_{1,1}+u_{3,3}) \quad & \tau_{13} &= \mu(u_{1,3}+u_{3,1}) \\ \tau_{33} &= -\mu(u_{1,1}+u_{2,2}) \quad & \tau_{23} &= \mu(u_{2,3}+u_{3,2}). \end{aligned} \tag{2}$$

Introducing eqns (2) into the equations of equilibrium

$$\tau_{ij,j} = 0, \tag{3}$$

we obtain

$$\begin{aligned} [-u_{2,1}+u_{1,2}]_{,2} + [u_{1,3}-u_{3,1}]_{,3} &= 0 \\ [-u_{1,2}+u_{2,1}]_{,1} + [u_{2,3}-u_{3,2}]_{,3} &= 0 \\ [-u_{1,3}+u_{3,1}]_{,1} + [u_{3,2}-u_{2,3}]_{,2} &= 0 \end{aligned} \tag{4}$$

or

$$\operatorname{rot}\operatorname{rot}\mathbf{u} = 0 \tag{4a}$$

which integrates to rot $\mathbf{u}$ = grad ϕ, ϕ being a scalar function.

The traction at the interface between inclusion and matrix is

$$T_i^n = \tau_{ij}n_j = \lambda n_i \nabla\cdot\mathbf{u} + \mu(u_{i,j}+u_{j,i})n_j \tag{5}$$

or, by introducing

$$\omega = \frac{\lambda+\mu}{\mu} = \frac{1}{1-2v} = \frac{2}{k-1}$$

(with k denoting the Kolosov constant),

$$T_i^n = \frac{T_i^n}{\mu} = \frac{\tau_{ij}n_j}{\mu} = (\omega-1)n_i\nabla\cdot\mathbf{u} + (u_{i,j}+u_{j,i})n_j. \tag{5a}$$

By algebraic manipulation eqn (5a) yields

$$\begin{aligned} T_1^n &= (\omega-1)n_1\nabla\cdot\mathbf{u} + (u_{1,k}+u_{k,1})n_k \\ &= (\omega+1)n_1\nabla\cdot\mathbf{u} - 2n_1\nabla\cdot\mathbf{u} + 2u_{1,1} + \sum_{k=2}^{3}(u_{1,k}+u_{k,1})n_k \\ &= (\omega+1)n_1\nabla\cdot\mathbf{u} - 2n_1(u_{2,2}+u_{3,3}) + \sum_{k=2}^{3}(u_{1,k}+u_{k,1})n_k \\ &= (\omega+1)n_1\nabla\cdot\mathbf{u} + 2(u_{2,2}(-n_1)+u_{2,1}n_2) \\ &\quad + 2(u_{3,3}(-n_1)+u_{3,1}n_3) + (u_{1,2}-u_{2,1})n_2 + (u_{1,3}-u_{3,1})n_3 \end{aligned} \tag{5b$_1$}$$

$$T_2^n = (\omega+1)\,n_2\,\nabla\cdot\mathbf{u}+2(u_{1,1}(-n_2)+u_{1,2}n_1)$$
$$+2(u_{3,3}(-n_2)+u_{3,2}n_3)+(u_{2,1}-u_{1,2})n_1+(u_{2,3}-u_{3,2})n_3 \quad (5b_2)$$

$$T_3^n = (\omega+1)\,n_3\,\nabla\cdot\mathbf{u}+2(u_{1,1}(-n_3)+(u_{1,3}n_1))+2(u_{2,2}(-n_3)$$
$$+u_{2,3}n_2)+(u_{3,1}-u_{1,3})n_1+(u_{3,2}-u_{2,3})n_2. \quad (5b_3)$$

This formula can be written in a vector form as

$$T_i^n = (\omega+1)n_i\,\mathrm{div}\,\mathbf{u}+2\,T_{\tau i}(\mathbf{u})-(\mathrm{rot}\,\mathbf{u}\times\mathbf{n})_i, \quad (5c)$$

where

$$T_{\tau 1}(\mathbf{u}) \overset{\text{def}}{=} \left(\frac{\partial u_2}{\partial x_1}n_2-\frac{\partial u_2}{\partial x_2}n_1\right)+\left(\frac{\partial u_3}{\partial x_1}n_3-\frac{\partial u_3}{\partial x_3}n_1\right)$$

$$T_{\tau 2}(\mathbf{u}) \overset{\text{def}}{=} \left(\frac{\partial u_1}{\partial x_1}(-n_2)+\frac{\partial u_1}{\partial x_2}n_1\right)+\left(\frac{\partial u_3}{\partial x_2}n_3-\frac{\partial u_3}{\partial x_3}n_2\right)$$

$$T_{\tau 3}(\mathbf{u}) \overset{\text{def}}{=} \left(\frac{\partial u_1}{\partial x_1}(-n_3)+\frac{\partial u_1}{\partial x_3}n_1\right)+\left(\frac{\partial u_2}{\partial x_2}(-n_3)+\frac{\partial u_2}{\partial x_3}n_2\right). \quad (5d)$$

Note that eqn (5c) differs from Kupradze (1979, p. 58) only by the differential operator $T_\tau(\mathbf{u})$ acting on a tangential plane of the surface. The last terms in eqns (5b) can be identified as $\mathrm{rot}\,\mathbf{u}\times\mathbf{n}$ which is a tangential component of the traction.

The two middle terms in eqns (5b) are the components of a vector which in general can be decomposed in both the normal and the tangential directions. Thus, setting $\omega\to-1$ leaves a traction in both the normal and tangential directions. However, in the case of a rigid inclusion with rigid body displacements

$$\mathbf{u} = \mathbf{\Omega}\times\mathbf{r}+\mathbf{b},$$

the two middle terms of eqns (5b) take the form

$$2\mathbf{\Omega}\times\mathbf{n}$$

and eqn (5c) is written as

$$\overline{\mathbf{T}^n} = (\omega+1)\,\mathbf{n}\,\nabla\cdot\mathbf{u}+2\mathbf{\Omega}\times\mathbf{n}-\mathrm{rot}\,\mathbf{u}\times\mathbf{n} \quad (6)$$

where the first term is along the normal and depends on Poisson's ratio, while the last two terms are on the tangential plane (shear component of the traction). Evaluating eqn (6) as $\omega\to-1$ (or equivalently $\lambda+2\mu\to 0$, or $k\to-1$), we obtain

$$\mathbf{T}^n = 2\mu\,(\mathbf{\Omega}-\boldsymbol{\omega})|_{k=-1}\times\mathbf{n} \quad (7)$$

where $\boldsymbol{\omega}\equiv\frac{1}{2}\,\mathrm{rot}\,\mathbf{u}$ and the above *constitutes the three-dimensional generalization* of the tangential traction $T = 2\mu\,(\Omega-\omega)|_{k=-1}$ in Markenscoff (1993). It should be noted here that in eqn (6) of Markenscoff (1993), the sign of ω_0 should be reversed for positive counter-clockwise rotation so that $u_{1,2}-u_{2,1} = -2\omega_0$. We note that the constant ω_0 depends only on the boundary data. For the evaluation of ω_0 in terms of the boundary data we refer to the Appendix. Actually, regarding the two-dimensional problem, the traction on the cavities may be made to vanish if one scalar parameter of the problem is properly chosen; for instance, by superposing the solution of a concentrated couple applied on the inclusion we can make $(\Omega - \omega_0)|_{k=-1} = 0$

The traction boundary value problem of three-dimensional elasticity satisfies the equilibrium equations (3) as well as the Beltrami–Mitchell compatibility conditions with body forces $\mathbf{F}$:

$$\nabla^2 \tau_{ij} + \frac{1}{1+\nu} \tau_{kk,ij} = -\frac{\nu}{1-\nu} \delta_{ij} \nabla \cdot \mathbf{F} - (F_{i,j} + F_{j,i}), \tag{8}$$

and if the stress field is such that the body forces are zero, and

$$\frac{\partial^2}{\partial x_i \partial x_j} \tau_{kk} = 0, \tag{9}$$

then it follows from eqn (8) that the Beltrami–Mitchell conditions are independent of the Poisson's ratio, and so is the solution of the traction boundary-value problem in this case. In this restricted case where the dilation is a constant or a linear function of x_i, the equivalence between cavities and rigid inclusions goes through, since the limit of stress field of the inclusion problem, if it exists, will satisfy equilibrium (3), compatibility (8) and boundary conditions (7), and by the uniqueness theorem will be unique, and will constitute the solution of the cavity problem loaded by shear tractions (7). If the dilatation is not constant or a linear function of the position, then the Poisson's ratio dependent term in the compatibility condition may be considered as a fictitious equivalent body force, and the correspondence may be thought of in this sense.

Example of correspondence between cavities and rigid inclusions in three-dimensions

Let us consider the problem of a spherical rigid inclusion of radius a at the center of a sphere of radius b, loaded by (a compressive) pressure $-p_0$ at $R = b$. The solution to this problem is the following (Sokolnikoff, 1956, p. 344)

$$\tau_{RR} = (3\lambda + 2\mu) \frac{(-p_0)}{\left(\dfrac{3\lambda+2\mu}{a^3} + \dfrac{4\mu}{b^3}\right)} \frac{1}{a^3} - \frac{4\mu}{R^3} \frac{p_0}{\left(\dfrac{3\lambda+2\mu}{a^3} + \dfrac{4\mu}{b^3}\right)}. \tag{10}$$

By setting $\omega = -1$ (or $k = -1$ or $\lambda + 2\mu = 0$), we obtain

$$\tau_{RR} = \frac{-p_0}{\left(\dfrac{1}{a^3} - \dfrac{1}{b^3}\right)} \frac{1}{a^3} + \frac{p_0}{\left(\dfrac{1}{a^3} - \dfrac{1}{b^3}\right)} \frac{1}{R^3} \tag{11}$$

which coincides with the solution of the cavity problem with the $R = a$ boundary stress free [since the right-hand side of eqn (7) is zero due to symmetry], and the outer boundary $R = b$ loaded with pressure $-p_0$ (Sokolnikoff, 1956, p. 345).

Indeed, in this case the dilation is constant, since

$$\tau_{\theta\theta} = \tau_{\phi\phi} = \frac{-p_0}{\left(\dfrac{1}{a^3} - \dfrac{1}{b^3}\right) a^3} - \frac{1}{2} \frac{p_0}{\left(\dfrac{1}{a^3} - \dfrac{1}{b^3}\right) R^3} \tag{12}$$

and the solution of the cavity problem is independent of Poisson's ratio. It can easily be seen that the stresses $\tau_{\theta\theta}$, $\tau_{\phi\phi}$ of the inclusion problem yield eqn (12) in the limit as $\omega \to -1$.

ON THE EXISTENCE OF THE LIMIT AS $k \to -1$

We now examine conditions under which the limit of the three-dimensional inclusion problem exists as $\omega \to -1$ (or $k \to -1$). We can prove existence of the limit, with the proof presented below in case a smooth inclusion is an unbounded domain. Let us consider the equations of linear elasticity for an infinite solid containing an inclusion; then we have a boundary-value problem of the type:

$$\begin{aligned} &\Delta \mathbf{u} + \omega \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \quad && x \in D \\ &\mathbf{u}|_\Gamma = \mathbf{u}_0(x) && x \in \Gamma = \partial D \end{aligned} \tag{13}$$

where, for a rigid inclusion, the displacement is one of a rigid body motion, i.e. $\mathbf{u}_0(x) = \mathbf{\Omega} \times \mathbf{r} + \mathbf{b}, x \in \Gamma$. We also assume that $\mathbf{u}_0(x)$ is a harmonic vector in D with a bounded Dirichlet integral

$$D(\mathbf{u}) \overset{\text{def}}{\equiv} \left(\sum_{i,j=1}^{3} \sum \int_D |\partial_j u_i|^2 \, \mathrm{d}x \right)^{1/2}.$$

The solution of eqn (13) is sought in the class of functions with bounded Dirichlet integral. Noting that the operator $\Delta \equiv \operatorname{grad} \operatorname{div} - \operatorname{rot}^2$, we rewrite eqn (13) in the form

$$\begin{aligned} &(1+\omega) \Delta \mathbf{u} + \omega \operatorname{rot}^2 \mathbf{u} = 0 \quad && x \in D \\ &\mathbf{u}|_\Gamma = \mathbf{u}_0 && x \in \Gamma. \end{aligned} \tag{14}$$

In order to investigate the behavior of the solution of eqn (14) in the vicinity of $\omega = -1$, we may make use of some results obtained in the context of the Cossérat spectrum analysis of the equations of elasticity. In the context of their research on the Cossérat spectrum (Cossérat, 1898; Mikhlin, 1970; Mazyá and Mikhlin, 1970), proved that the solution of problem (14) may be scaled in terms of the parameter $\varepsilon = 1 + \omega/\omega$ to

$$\begin{gathered} \Delta \mathbf{w} + \operatorname{rot} \mathbf{q} = 0, \quad \mathbf{w} = \varepsilon \mathbf{u}, \quad \operatorname{rot} \mathbf{u} = \mathbf{q}, \quad \varepsilon = \frac{1+\omega}{\omega} \\ \mathbf{w}|_r = \varepsilon \mathbf{u}_0, \quad \operatorname{div} \mathbf{q} = 0, \quad (\mathbf{q}, \mathbf{n})|_\Gamma = q_0, \quad \int_\Gamma q_0 \, \mathrm{d}\Gamma = 0 \end{gathered} \tag{15}$$

where w and q admit solution in the form of convergent power series in the vicinity of $\varepsilon = 0$ (or $\omega = -1$).

$$\mathbf{w} = \sum_{m=0}^{\infty} \varepsilon^m \mathbf{w}^{(m)}, \quad \mathbf{q} = \sum_{m=v}^{\infty} \varepsilon^m \mathbf{q}^{(m)} \tag{16}$$

and q_0 may be expressed in terms of the displacement boundary conditions since

$$q_0 = (\mathbf{q}, \mathbf{n})|_\Gamma$$

only contains the derivatives of the displacement along the tangential directions. For instance, in spherical coordinates,

$$q_0 = \left[\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (u_\phi \sin\theta) - \frac{1}{r \sin\theta} \frac{\partial u_\theta}{\partial \phi} \right].$$

As a result, from eqn (15) follows a sequence of boundary value problems for the determination of the coefficients $\mathbf{w}^{(m)}$, $\mathbf{q}^{(m)}$, which are for $m = 0$:

$$\Delta \mathbf{w}^{(0)} + \operatorname{rot} \mathbf{q}^{(0)} = 0, \quad \operatorname{rot} \mathbf{w}^{(0)} = 0$$

$$\mathbf{w}^{(0)}|_r = 0, \quad \operatorname{div} \mathbf{q}^{(0)} = 0, \quad (\mathbf{q}^{(0)}, \mathbf{n})|_\Gamma = q_0, \quad \int_\Gamma q_0 \, \mathrm{d}r = 0; \tag{17}$$

for $m = 1$:

$$\Delta \mathbf{w}^{(1)} + \operatorname{rot} \mathbf{q}^{(1)} = 0, \quad \mathbf{q}^{(0)} = \operatorname{rot} \mathbf{w}^{(1)}$$

$$\mathbf{w}^{(1)}|_\Gamma = \mathbf{u}_0, \quad \operatorname{div} \mathbf{q}^{(1)} = 0, \quad (\mathbf{q}^{(1)}, \mathbf{n})|_\Gamma = 0. \tag{18}$$

Let us multiply the three components of the first equation of (17) by $w_1^{(0)}, w_2^{(0)}, w_3^{(0)}$ respectively and integrate in the volume by parts so that we obtain

$$\int_D \{[\Delta w_1^{(0)} + (\operatorname{rot} \mathbf{q}^{(0)})_1] w_1^{(0)} + [\Delta w_2^{(0)} + (\operatorname{rot} \mathbf{q}^0)_2] w_2^{(0)} + [\Delta w_3^{(0)} + (\operatorname{rot} \mathbf{q}^{(0)})_3] w_3^{(0)}\} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3$$

$$= -\int_D \{|\nabla w_1^{(0)}|^2 + |\nabla w_2^{(0)}|^2 + |\nabla w_3^{(0)}|^2\} \, \mathrm{d}x + \int_D (\mathbf{q}^{(0)}, \operatorname{rot} \mathbf{w}^{(0)}) \, \mathrm{d}x = 0. \tag{19}$$

However, the last integral in eqn (19) vanishes so that, from eqns (19) and (17), it follows that $w_1^{(0)}, w_2^{(0)}, w_3^{(0)}$ are constants taking their boundary value, which is 0. Thus, the solution of problem (17) is

$$w_1^{(0)} = w_2^{(0)} = w_2^{(0)} = 0$$

$$q^{(0)} = \operatorname{grad} \phi \tag{20}$$

where ϕ is the solution of

$$\Delta \phi = 0 \qquad \text{in } D$$

$$\frac{\partial \phi}{\partial n} = q_0, \quad \int q_0 \, \mathrm{d}\Gamma = 0 \quad \text{on } \Gamma \tag{21}$$

Then, according to Mazyá and Mikhlin (1970), problem (18) has a solution and consequently the limit (as $\omega \to -1$) of problem (14) exists, and is $\lim_{\omega \to -1} u(x) = w^{(1)}(x)$.

Sufficient smoothness of the boundary of the inclusion was assumed which does not cover the case of an anticrack. Also, the displacement on the boundary of the inclusion was assumed to be of O(1), which does not cover the case when the rotation of the inclusion is of order ε^{-1}.

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APPENDIX

In two-dimensional elasticity, we found $\omega_0|_{k=-1}$ for the displacement and traction boundary value problem containing an inclusion. If the displacements on the exterior boundary (Fig. A1) are known, then it's easy to see by Stokes' theorem that

$$\omega_0 = \frac{1}{2}\frac{\int_{\Gamma_i}(\mathbf{u}_0,\mathbf{s})\,\mathrm{d}\Gamma + \int_{\Gamma_e}(\mathbf{u}_0',\mathbf{s})\,\mathrm{d}\Gamma}{\mathrm{mes}\,(D)}. \tag{A1}$$

If the tractions on the exterior boundary are known, then

$$\omega_0 = \frac{1}{2}\frac{\frac{1}{2\mu}M - \int_{\Gamma_i}(\mathbf{u}_0,\mathbf{s})\,\mathrm{d}\Gamma}{\mathrm{mes}\,(D_0)} \tag{A2}$$

where

$$M = \int_{\Gamma_e}(T_2x_1 - T_1x_2)\,\mathrm{d}\Gamma$$

is the moment of the traction $\mathbf{T}$ acting to the external boundary, mes (D_0) is the area of the inclusion.

In order to prove the formula (A2), we find the displacements from the boundary conditions [see Markenscoff (1993)]

$$\frac{1}{\mu}T_1 = -2\frac{\mathrm{d}u_2}{\mathrm{d}s} - 2\omega_0 n_2,\quad \frac{1}{\mu}T_2 = 2\frac{\mathrm{d}u_1}{\mathrm{d}s} + 2\omega_0 n_1 \tag{A3}$$

by integration of (A3) with respect to s where $s \in \Gamma_e$, we obtain

$$\frac{1}{\mu}\int_0^s T_1\,\mathrm{d}s + 2\omega_0\int_0^s n_2\,\mathrm{d}s + c_2 = -2u_2,\quad s\in\Gamma_i,$$

$$\frac{1}{\mu}\int_0^s T_2\,\mathrm{d}s - 2\omega_0\int_0^s n_1\,\mathrm{d}s - c_1 = 2u_1,\quad s\in\Gamma_e, \tag{A4}$$

where c_1 and c_2 are constants. It is easy to see, that

$$\int_0^s n_2\,\mathrm{d}s = -x_1,\quad \int_0^s n_1\,\mathrm{d}s = x_2.$$

Therefore, by substituting the displacements from (A4) into the formula

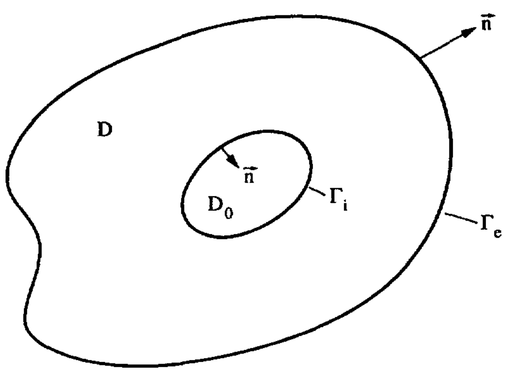

Fig. A1.

$$2\omega_0 \operatorname{mes}(D) = \int_D \operatorname{rot} \mathbf{u} \, dx = \int_{\Gamma_e} (\mathbf{u}, \mathbf{s}) \, d\Gamma + \int_{\Gamma_i} (\mathbf{u}_0, \mathbf{s}) \, d\Gamma, \tag{A5}$$

we find

$$2\omega_0 \operatorname{mes}(D) = \int_{\Gamma_i} (\mathbf{u}_0, \mathbf{s}) \, d\Gamma + \omega_0 \int_{\Gamma_e} (\mathbf{x}, \mathbf{n}) \, d\Gamma - \frac{1}{2\mu} \int_{\Gamma_e} \left[\left(\int_0^s T_1 \, ds + c_2 \right) n_1 + \left(\int_0^s T_2 \, ds - c_1 \right) n_2 \right] d\Gamma \tag{A6}$$

where the integral pertaining to the constants c_1 and c_2 vanishes for single valued displacements. Again noting that

$$n_1 = \frac{dx_2}{ds}, \quad n_2 = -\frac{dx_1}{ds}$$

and integrating by parts in the last integral of (A6), we have

$$2\omega_0 \operatorname{mes}(D) = \int_{\Gamma_i} (\mathbf{u}_0, \mathbf{s}) \, d\Gamma + \omega_0 \int_{\Gamma_e} (\mathbf{x}, \mathbf{n}) \, d\Gamma - \frac{1}{2\mu} M \tag{A7}$$

where

$$M \equiv \int_{\Gamma_e} (T_2 x_1 - T_1 x_2) \, d\Gamma.$$

Taking into consideration that

$$\int_{\Gamma_e} (\mathbf{x}, \mathbf{n}) \, d\Gamma = 2 \operatorname{mes}(DUD_0),$$

we obtain the formula (A2).